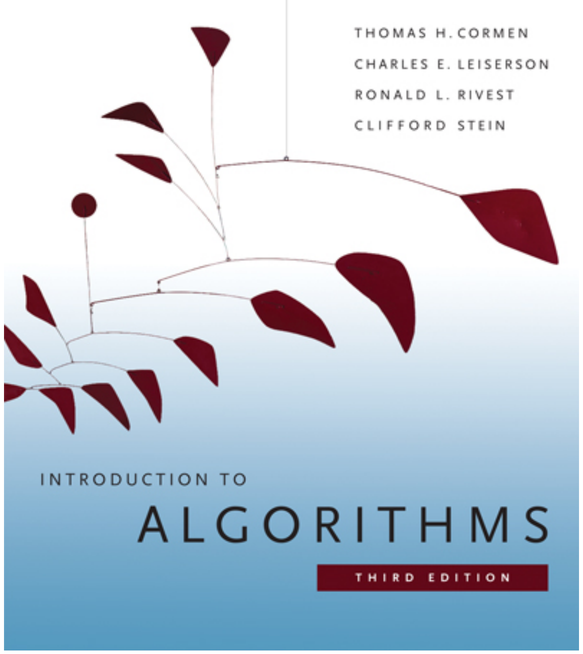
6.006- *Introduction to Algorithms*



Lecture 10

Prof. Constantinos Daskalakis

CLRS 8.1-8.4

Menu

- Show that $\Theta(n \lg n)$ is the best possible running time for a sorting algorithm.
- Design an algorithm that sorts in $\Theta(n)$ time.
- Hint: maybe the models are different ?

Comparison sort

All the sorting algorithms we have seen so far are ***comparison sorts***: only use comparisons to determine the relative order of elements.

- *E.g.*, merge sort, heapsort.

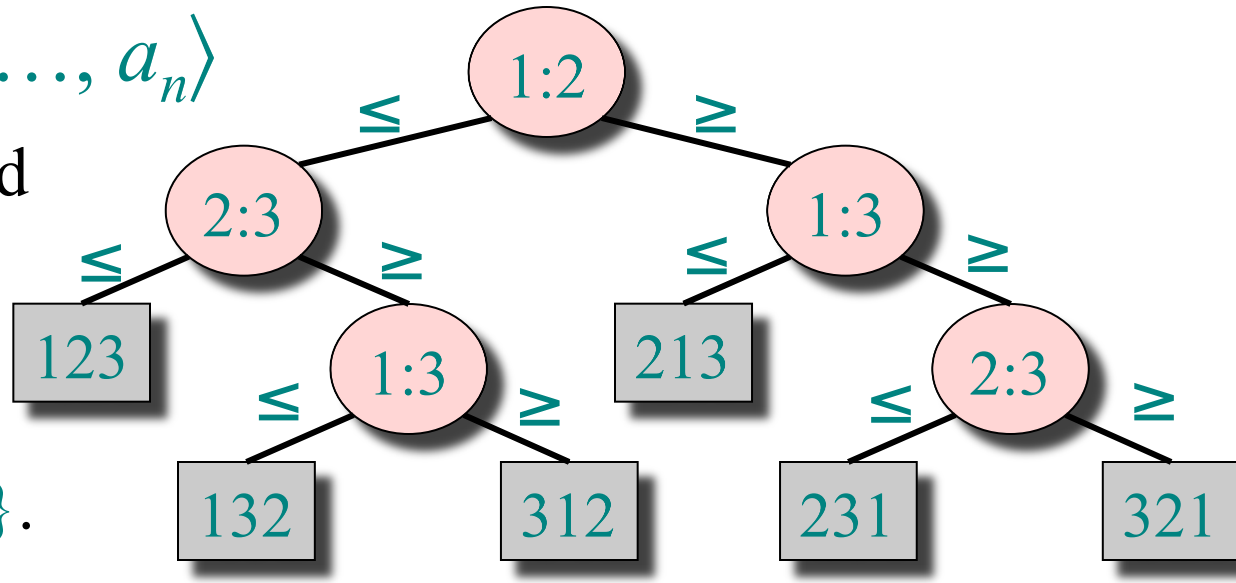
The best running time that we've seen for comparison sorting is $O(n \lg n)$.

Is $O(n \lg n)$ the best we can do?

Decision trees can help us answer this question.

Decision-tree

A recipe for sorting n numbers $\langle a_1, a_2, \ldots, a_n\rangle$



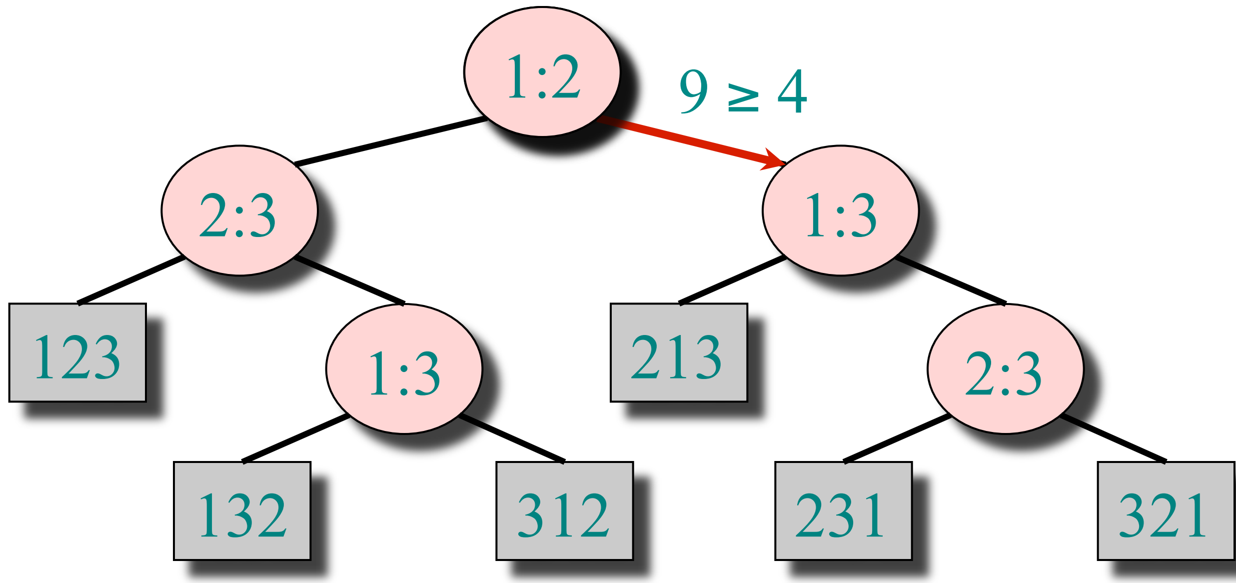
- Nodes are suggested comparisons: i:j means compare a_i to a_j, for $i, j \in \{1, 2, \ldots, n\}$.

- Branching direction depends on outcome of comparisons.

- Leaves are labeled with permutations corresponding to the outcome of the sorting.

Decision-tree example

Sort $\langle a_1, a_2, a_3 \rangle$
$= \langle\ 9, 4, 6\ \rangle$:



Each internal node is labeled $i{:}j$ for $i, j \in \{1, 2, \ldots, n\}$.

- The left subtree shows subsequent comparisons if $a_i \le a_j$.
- The right subtree shows subsequent comparisons if $a_i \ge a_j$.

Decision-tree example

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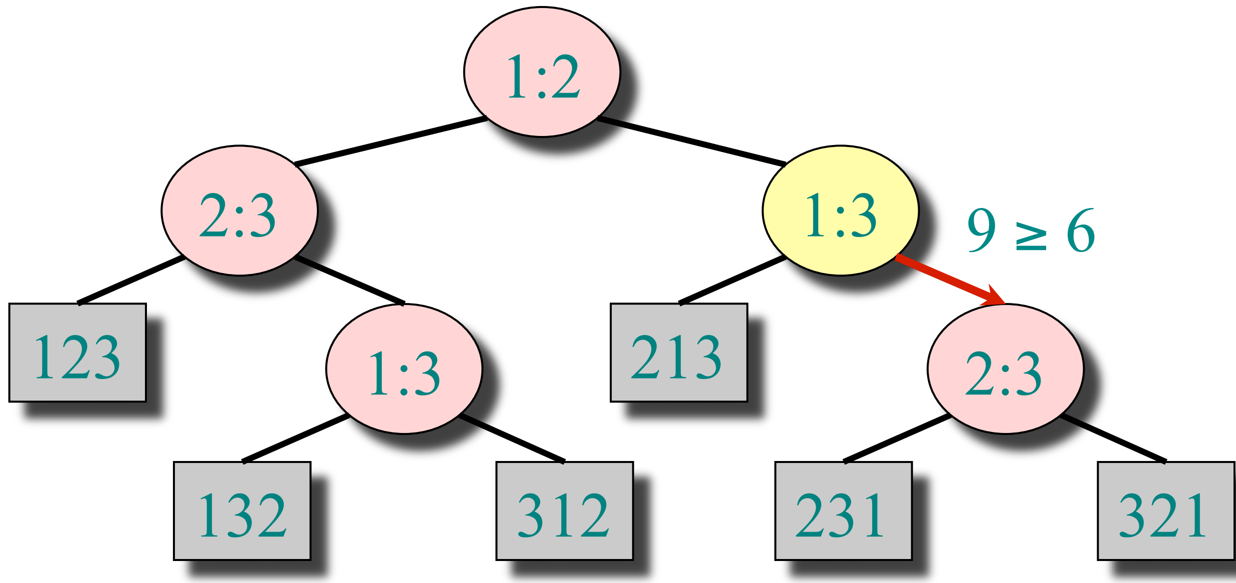
$= \langle\ 9, 4, 6\ \rangle$:



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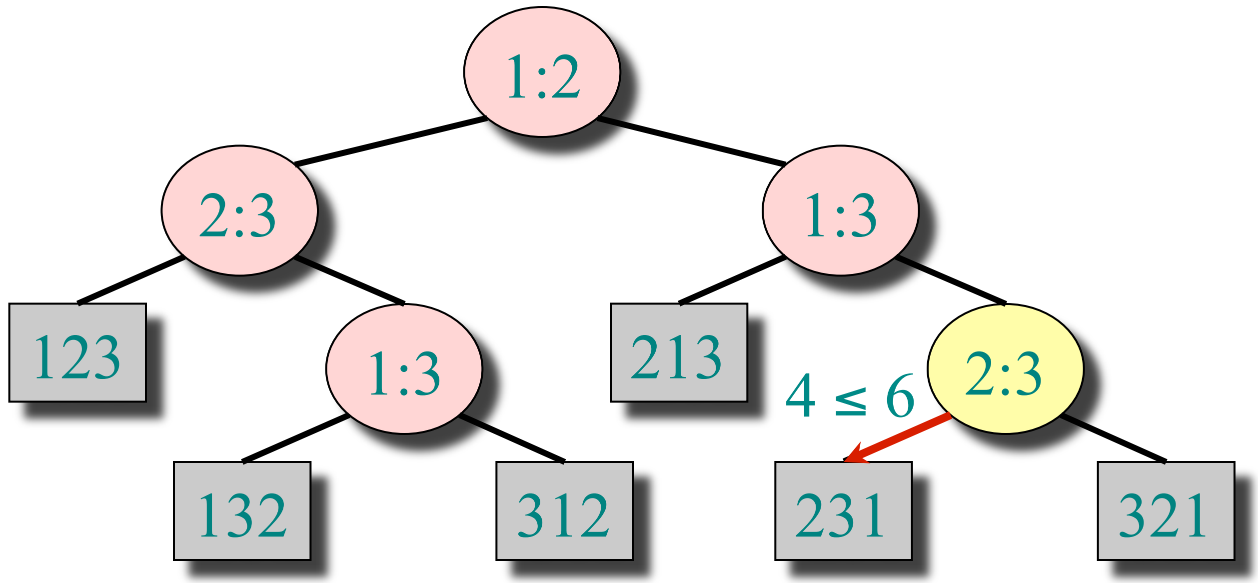

Each internal node is labeled $i{:}j$ for $i, j \in \{1, 2, \ldots, n\}$.

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Decision-tree example

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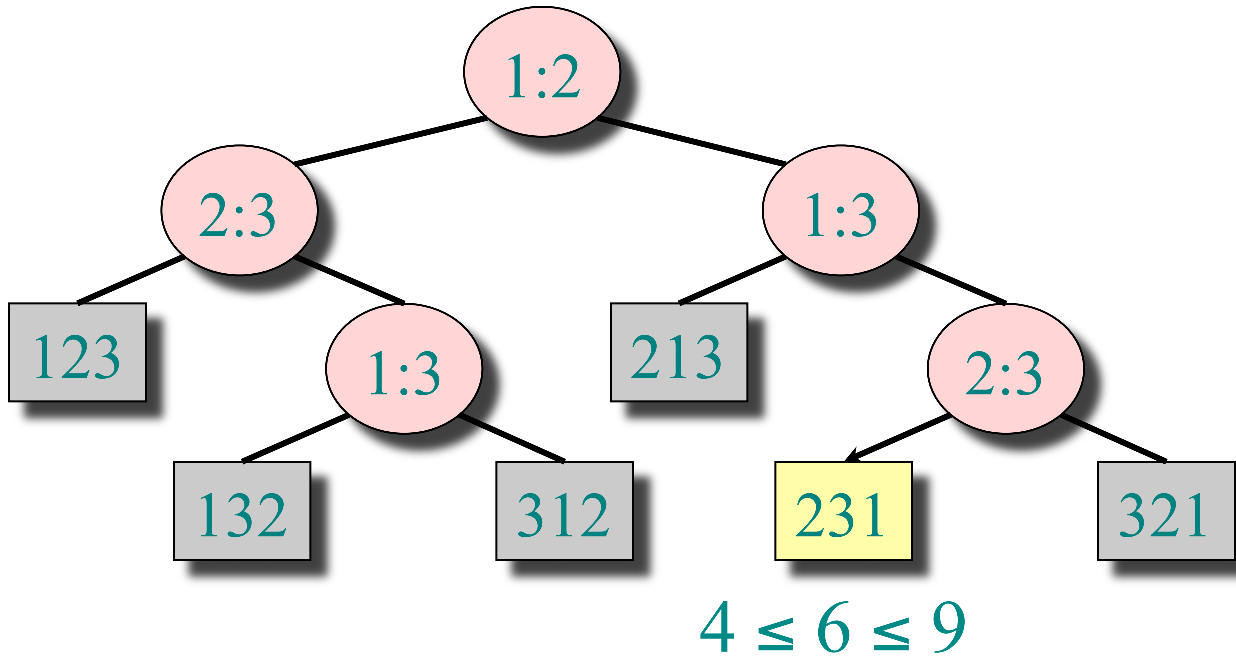


Each leaf contains a permutation $\langle \pi(1), \pi(2), \ldots, \pi(n) \rangle$ to indicate that the ordering $a_{\pi(1)} \leq a_{\pi(2)} \leq \cdots \leq a_{\pi(n)}$ has been established.

Decision-tree model

A decision tree can model the execution of any comparison sort:

- One tree for each input size n.
- A path from the root to the leaves of the tree represents a trace of comparisons that the algorithm may perform.
- The running time of the algorithm = the length of the path taken.
- Worst-case running time = height of tree.

Lower bound for decision-tree sorting

Theorem. Any decision tree that can sort n elements must have height $\Omega(n \lg n)$.

Proof. (Hint: how many leaves are there?)

- The tree must contain $\geq n!$ leaves, since there are $n!$ possible permutations
- A height-h binary tree has $\leq 2^h$ leaves
- Thus

$$
\begin{aligned}
2^h &\geq n! \\
h &\geq \lg(n!) && \text{(lg is mono. increasing)} \\
&\geq \lg((n/e)^n) && \text{(Stirling's formula)} \\
&= n \lg n - n \lg e \\
&= \Omega(n \lg n) .
\end{aligned}
$$

Sorting in linear time

Counting sort: No comparisons between elements.

- ***Input***: $A[1 \,.\,.\, n]$, where $A[j] \in \{1, 2, \ldots, k\}$.
- ***Output***: $B[1 \,.\,.\, n]$, a sorted permutation of A
- ***Auxiliary storage***: $C[1 \,.\,.\, k]$.

Counting sort

for $i \leftarrow 1$ **to** k
    do $C[i] \leftarrow 0$
for $j \leftarrow 1$ **to** n
    do $C[A[j]] \leftarrow C[A[j]] + 1$

store in C the frequencies of the different keys in A i.e. $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k
    do $C[i] \leftarrow C[i] + C[i-1]$

now C contains the cumulative frequencies of different keys in A, i.e. $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1
    do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$
        $C[A[j]] \leftarrow C[A[j]] - 1$

using cumulative frequencies build sorted permutation

Counting-sort example

	1	2	3	4	5
A:	4	1	3	4	3

B:					

one index for each possible key stored in A

	1	2	3	4
C:				

Loop 1: initialization

	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	**2**	**3**	**4**
C:	0	0	0	0

for $i \leftarrow 1$ **to** k

  do $C[i] \leftarrow 0$

Loop 2: count frequencies

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	0	0	0	1

B:					

for $j \leftarrow 1$ **to** n

  do $C[A[j]] \leftarrow C[A[j]] + 1$ $\quad \triangleright\ C[i] = |\{\text{key} = i\}|$

Loop 2: count frequencies

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	0	1

B:					

for $j \leftarrow 1$ **to** n

  do $C[A[j]] \leftarrow C[A[j]] + 1 \quad \triangleright\ C[i] = |\{\text{key} = i\}|$

Loop 2: count frequencies

	1	2	3	4	5
A:	4	1	3	4	3

	1	**2**	**3**	**4**
C:	1	0	1	1

B:					

for $j \leftarrow 1$ **to** n
  do $C[A[j]] \leftarrow C[A[j]] + 1 \quad \triangleright\ C[i] = |\{\text{key} = i\}|$

Loop 2: count frequencies

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	1	2

B:					

for $j \leftarrow 1$ **to** n

  do $C[A[j]] \leftarrow C[A[j]] + 1 \quad \triangleright\ C[i] = |\{\text{key} = i\}|$

Loop 2: count frequencies

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	2	2

B:					

for $j \leftarrow 1$ **to** n

  do $C[A[j]] \leftarrow C[A[j]] + 1 \quad \triangleright\ C[i] = |\{\text{key} = i\}|$

Loop 2: count frequencies

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	2	2

B:					

for $j \leftarrow 1$ **to** n
  do $C[A[j]] \leftarrow C[A[j]] + 1$ $\quad \triangleright\ C[i] = |\{\text{key} = i\}|$

[A parenthesis: a quick finish

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	2	2

B:					

Walk through frequency array an place the appropriate number of each key in output array…

A parenthesis: a quick finish

	1	2	3	4	5
A:	4	1	3	4	3
B:	1				

	1	2	3	4
C:	1	0	2	2

A parenthesis: a quick finish

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	2	2

B:	1				

A parenthesis: a quick finish

	1	2	3	4	5
A:	4	1	3	4	3
B:	1	3	3		

	1	2	3	4
C:	1	0	2	2

A parenthesis: a quick finish

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	2	2

B:	1	3	3	4	4

B is sorted!

but it is not “stably sorted”…]

Loop 2: count frequencies

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	2	2

B:					

for $j \leftarrow 1$ **to** n

  do $C[A[j]] \leftarrow C[A[j]] + 1$ $\quad \triangleright\ C[i] = |\{\text{key} = i\}|$

Loop 3: cumulative frequencies

	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	2	3	4
C:	1	0	2	2
C':	1	1	2	2

for $i \leftarrow 2$ **to** k
    do $C[i] \leftarrow C[i] + C[i-1]$ ▷ $C[i] = |\{\text{key} \leq i\}|$

Loop 3: cumulative frequencies

	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	**2**	**3**	**4**
C:	1	0	2	2
C':	1	1	3	2

for $i \leftarrow 2$ **to** k
  do $C[i] \leftarrow C[i] + C[i-1]$ $\triangleright$ $C[i] = |\{\text{key} \le i\}|$

Loop 3: cumulative frequencies

	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	2	3	4
C:	1	0	2	2
C':	1	1	3	5

for $i \leftarrow 2$ **to** k

  do $C[i] \leftarrow C[i] + C[i-1]$ $\triangleright$ $C[i] = |\{\text{key} \le i\}|$

Loop 4: permute elements of A

A:

1	2	3	4	5
4	1	3	4	3

C:

1	2	3	4
1	1	3	5

B:

for $j \leftarrow n$ **downto** 1
    do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$
        $C[A[j]] \leftarrow C[A[j]] - 1$

Loop 4: permute elements of A

	1	2	3	4	5
A:	4	1	3	4	3

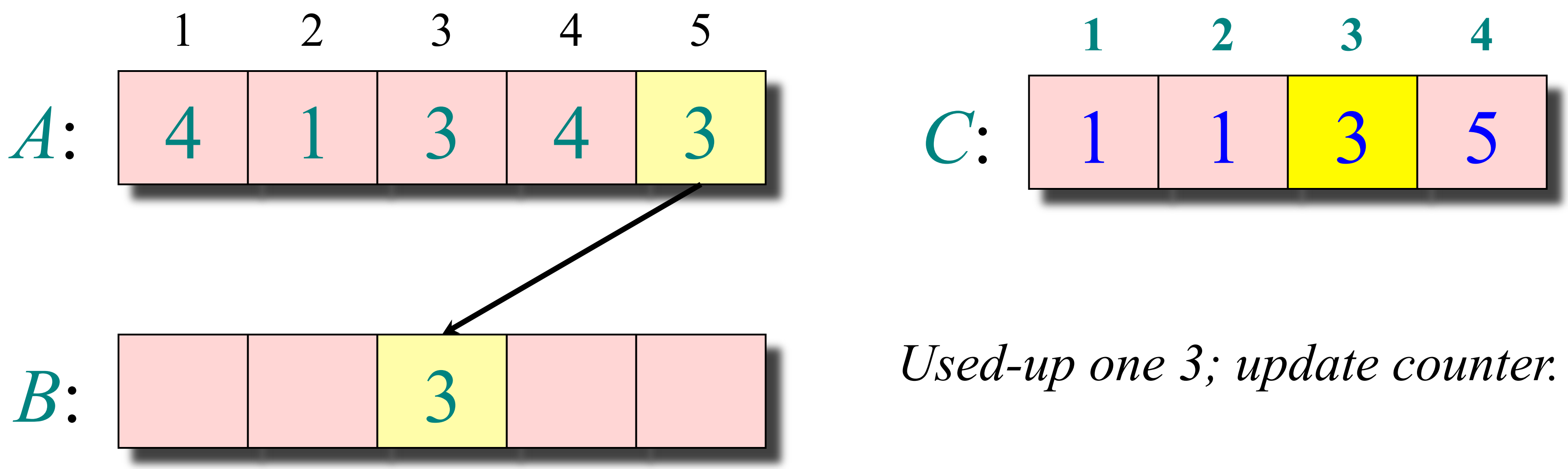
	1	2	3	4
C:	1	1	3	5

B:					

There are exactly 3 elements ≤A[5]; so where should I place A[5]?

for $j \leftarrow n$ **downto** 1
  do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$
    $C[A[j]] \leftarrow C[A[j]] - 1$

Loop 4: permute elements of A



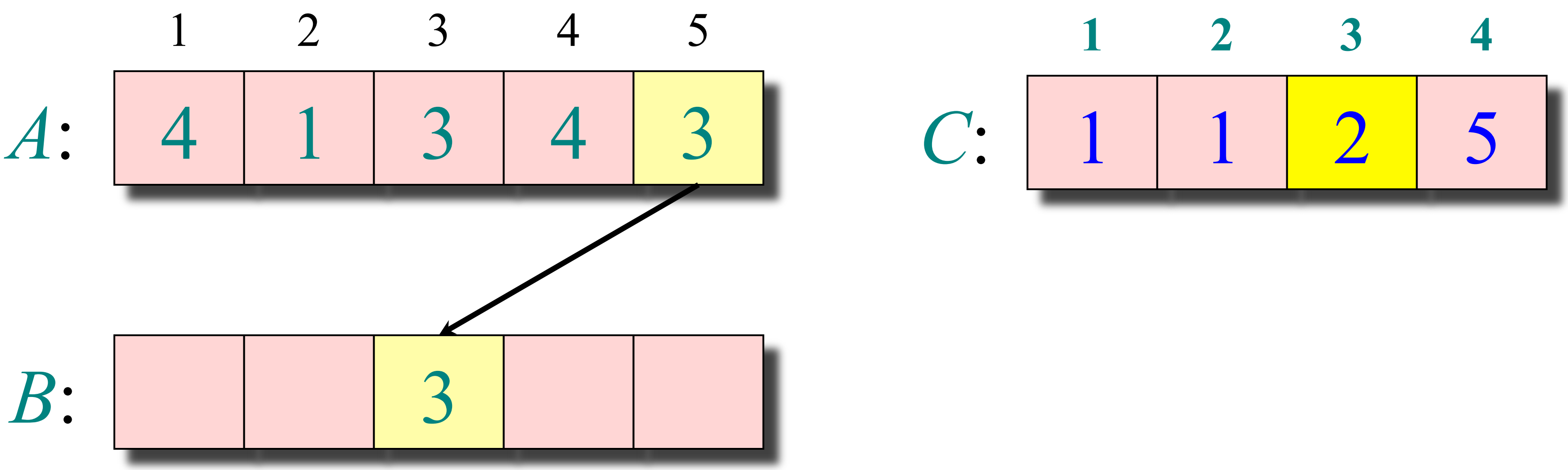
Used-up one 3; update counter.

for $j \leftarrow n$ **downto** 1

  do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$

    $C[A[j]] \leftarrow C[A[j]] - 1$

Loop 4: permute elements of A



for $j \leftarrow n$ **downto** 1

  do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$

    $C[A[j]] \leftarrow C[A[j]] - 1$

Loop 4: permute elements of A

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	1	2	5

B:			3		

for $j \leftarrow n$ **downto** 1

  do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$

    $C[A[j]] \leftarrow C[A[j]] - 1$

Loop 4: permute elements of A

	1	2	3	4	5
A:	4	1	3	4	3

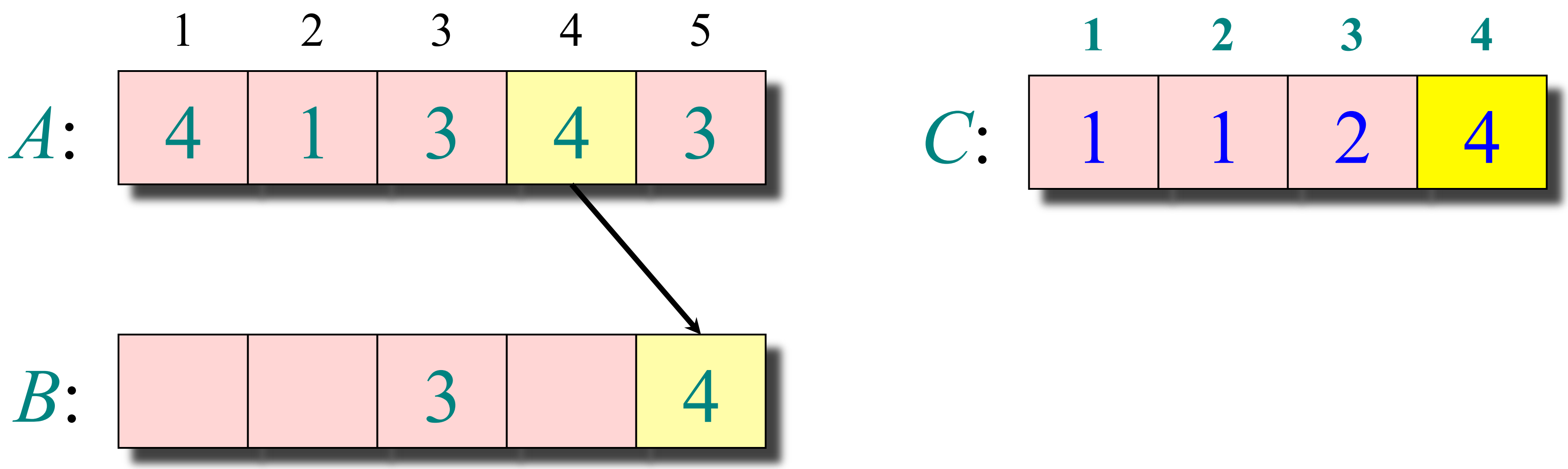
	1	2	3	4
C:	1	1	2	5

B:			3		

There are exactly 5 elements $\leq A[4]$, so where should I place $A[4]$?

for $j \leftarrow n$ **downto** 1
    do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$
        $C[A[j]] \leftarrow C[A[j]] - 1$

Loop 4: permute elements of A



for $j \leftarrow n$ **downto** 1
  do $B[C[A[j]]] \leftarrow A[j]$
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Loop 4: permute elements of A

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	1	2	4

B:			3		4

for $j \leftarrow n$ **downto** 1

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Loop 4: permute elements of A

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A:	4	1	3	4	3

	1	2	3	4
C:	1	1	2	4

B:			3		4

for $j \leftarrow n$ **downto** 1

  do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$

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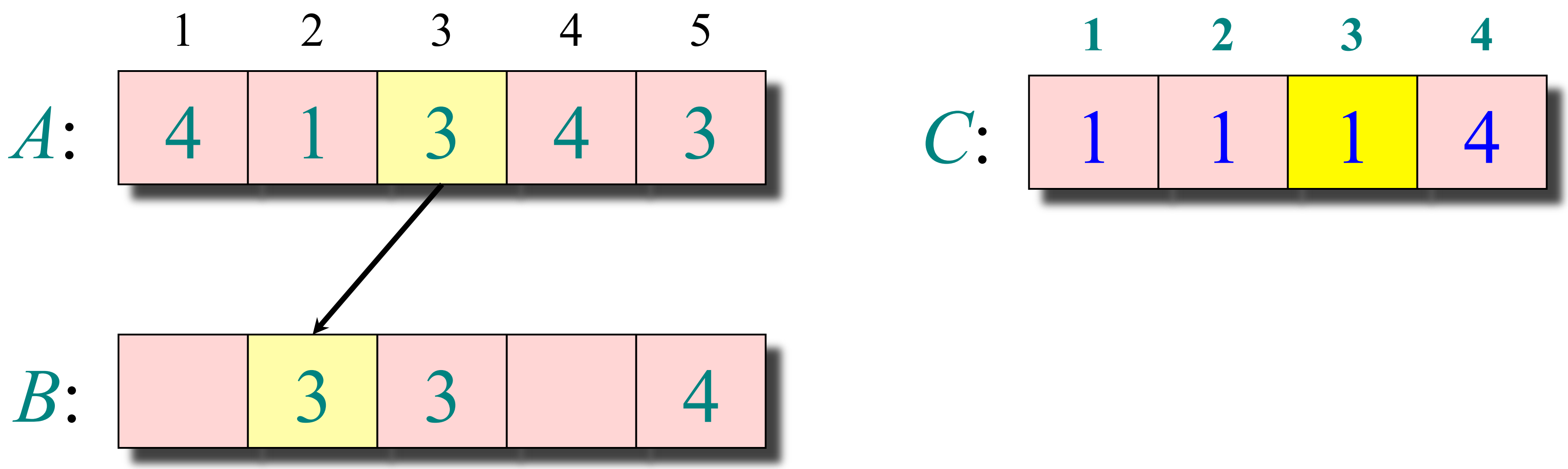
Loop 4: permute elements of A



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Loop 4: permute elements of A

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	1	1	4

B:		3	3		4

for $j \leftarrow n$ **downto** 1
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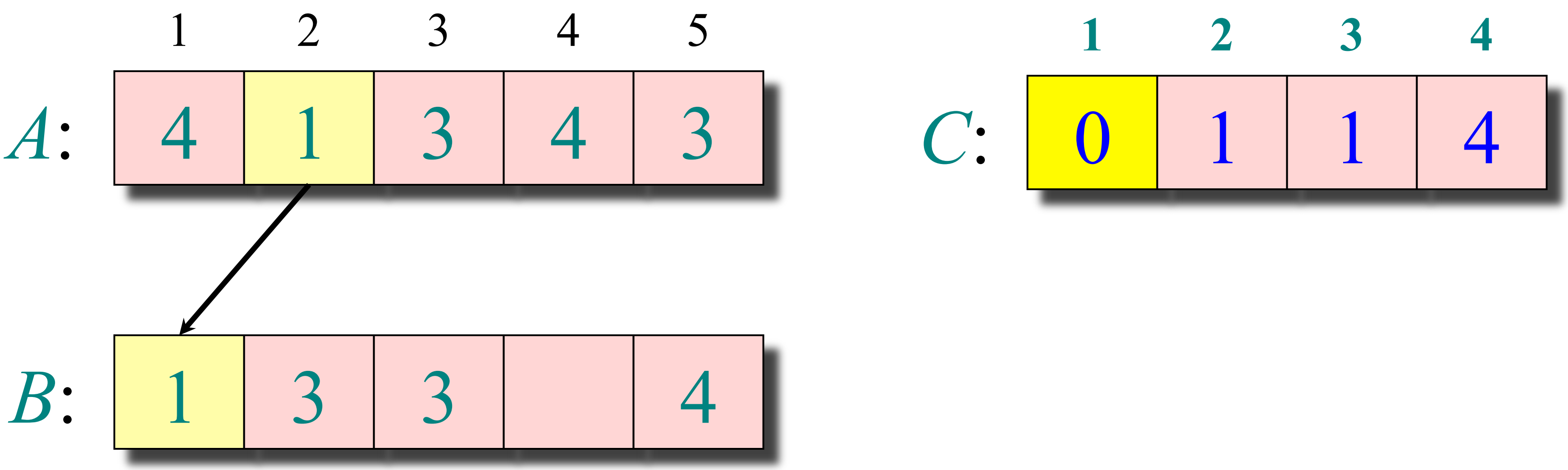
Loop 4: permute elements of A



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Loop 4: permute elements of A

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	0	1	1	4

B:	1	3	3		4

for $j \leftarrow n$ **downto** 1
  do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$
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Loop 4: permute elements of A

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	0	1	1	4

B:	1	3	3		4

for $j \leftarrow n$ **downto** 1

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for $j \leftarrow n$ **downto** 1

  do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$

    $C[A[j]] \leftarrow C[A[j]] - 1$

Analysis

$\Theta(k)$	**for** $i \leftarrow 1$ **to** k **do** $C[i] \leftarrow 0$
$\Theta(n)$	**for** $j \leftarrow 1$ **to** n **do** $C[A[j]] \leftarrow C[A[j]] + 1$
$\Theta(k)$	**for** $i \leftarrow 2$ **to** k **do** $C[i] \leftarrow C[i] + C[i-1]$

$\Theta(n)$	**for** $j \leftarrow n$ **downto** 1
	  do $B[C[A[j]]] \leftarrow \mathrm{A}[j]$
	  $C[A[j]] \leftarrow C[A[j]] - 1$

$\Theta(n + k)$	

Running time

If $k = O(n)$, then counting sort takes $\Theta(n)$ time.

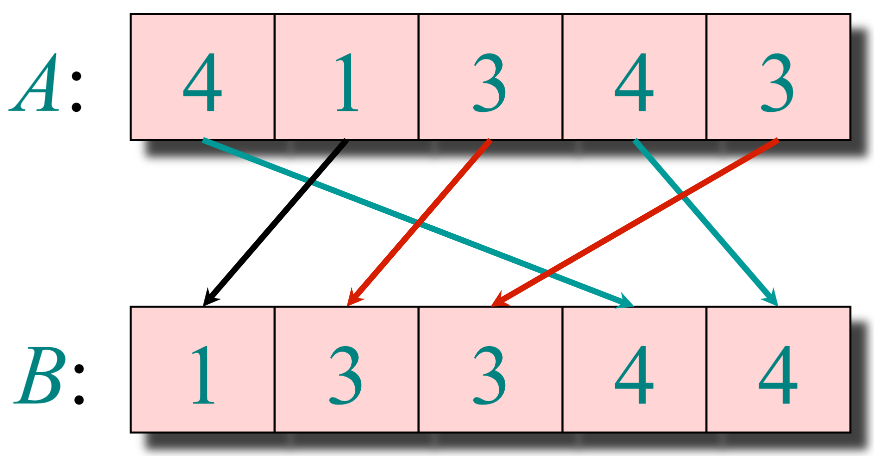
- But, sorting takes $\Omega(n \lg n)$ time!
- Where's the fallacy?

Answer:

- ***Comparison sorting*** takes $\Omega(n \lg n)$ time.
- Counting sort is not a ***comparison sort***.
- In fact, not a single comparison between elements occurs!

Stable sorting

Counting sort is a ***stable*** sort: it preserves the input order among equal elements.



Radix sort

- ***Origin***: Herman Hollerith's card-sorting machine for the 1890 U.S. Census. (See Appendix .)
- Digit-by-digit sort.
- Hollerith's original (bad) idea: sort on most-significant digit first.
- Good idea: Sort on ***least-significant digit first*** with auxiliary ***stable*** sort.

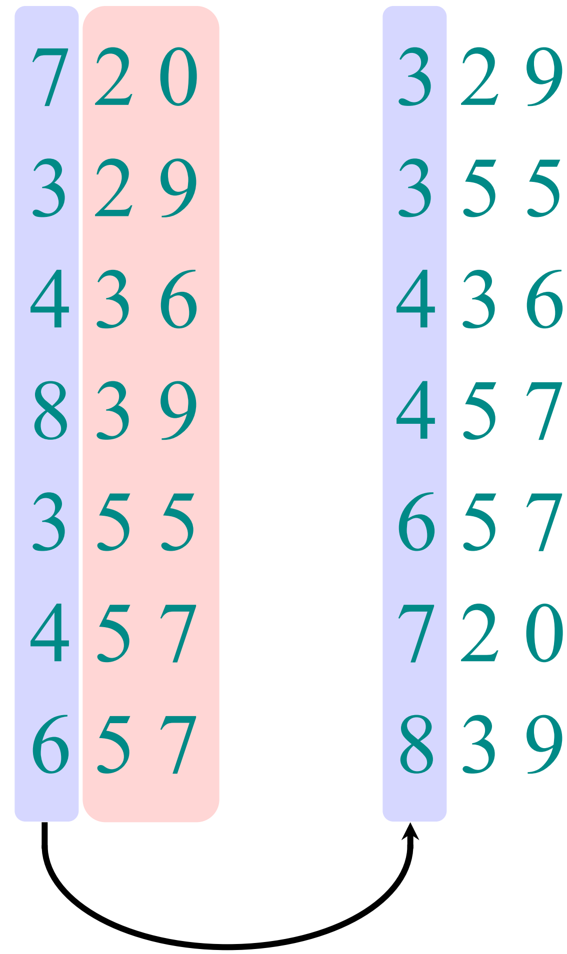
Operation of radix sort

3 2 9	7 2 0	7 2 0	3 2 9
4 5 7	3 5 5	3 2 9	3 5 5
6 5 7	4 3 6	4 3 6	4 3 6
8 3 9	4 5 7	8 3 9	4 5 7
4 3 6	6 5 7	3 5 5	6 5 7
7 2 0	3 2 9	4 5 7	7 2 0
3 5 5	8 3 9	6 5 7	8 3 9

Correctness of radix sort

Induction on digit position

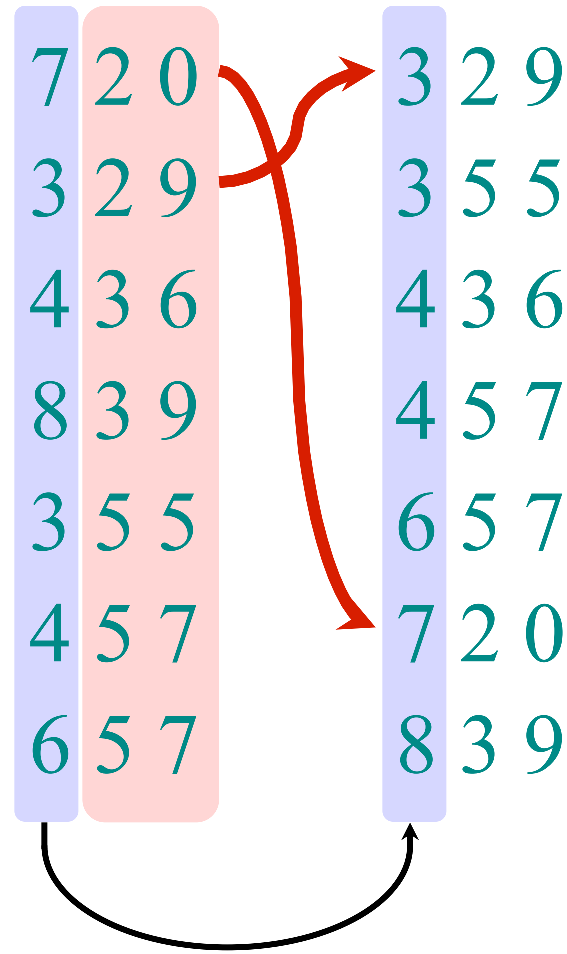
- Assume that the numbers are sorted by their low-order $t - 1$ digits.
- Sort on digit t



Correctness of radix sort

Induction on digit position

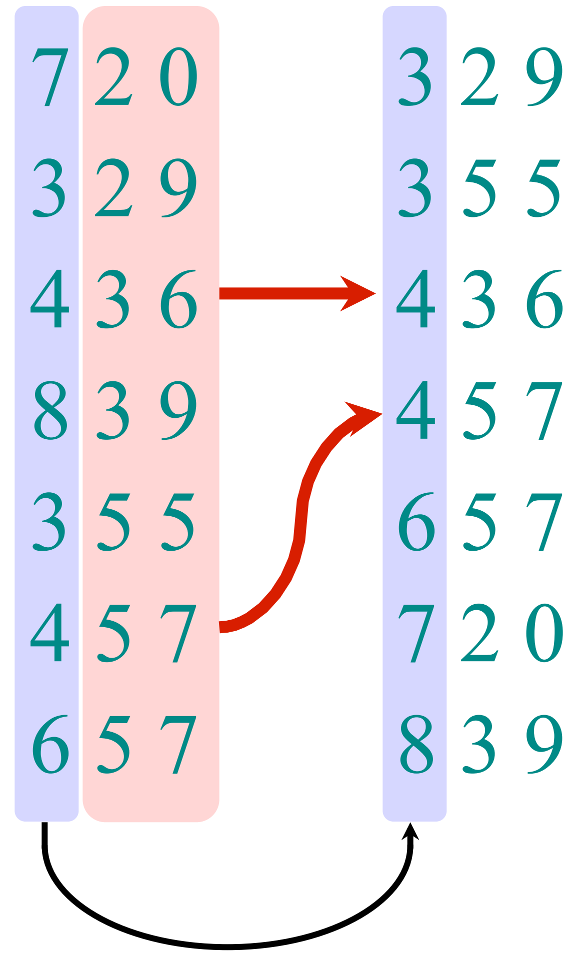
- Assume that the numbers are sorted by their low-order $t - 1$ digits.
- Sort on digit t
  - Two numbers that differ in digit t are correctly sorted.



Correctness of radix sort

Induction on digit position

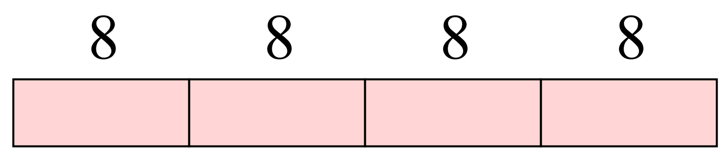
- Assume that the numbers are sorted by their low-order $t - 1$ digits.
- Sort on digit t
  - Two numbers that differ in digit t are correctly sorted.
  - Two numbers equal in digit t are put in the same order as the input ⇒ correct order.



Runtime Analysis of radix sort

- Assume counting sort is the auxiliary stable sort.
- Sort n computer words of b bits each.
- Each word can be viewed as having b/r base-2^r digits.

Example: 32-bit word



- If each b-bit word is broken into r-bit pieces, each pass of counting sort takes $\Theta(n + 2^r)$ time.
- Setting $r=\log n$ gives $\Theta(n)$ time per pass, or

$\Theta(n\ b/\log n)$ total